1. Span and Linear Independence

Let V be a vector space with underlying field $\mathbb{F}$. Recall that a vector space V is a module over a ring - in this case, the ring is the field $\mathbb{F}$.

Definition 1. *Call a subset $E \subset V$ **independent** if no finite nontrivial linear combination of vectors is equal to the zero vector. That is, we cannot find $e_1, ..., e_n \in E$ and $f_1, ..., f_n \in \mathbb{F}$, not all zero, such that $f_1e_1 + ... + f_ne_n = \mathbf{0}$.*

Definition 2. *If we have a set $E \subset V$, we define the **span** of E to be the set of all $f_1e_1 + ... + f_ne_n$ for which $f_1, ..., f_n \in \mathbb{F}$ and $e_1, ..., e_n \in E$; that is, the set of all linear combinations of elements of E.*

We will define a **basis** of a vector space V to be any set E whose span is V, and is independent. We claim that every vector space has a basis. The proof of this will rely upon Zorn's Lemma.

Theorem 1. *Every vector space has a basis.*

Proof.

- Consider the poset of all independent subsets of V ordered by inclusion.
- We will first show that there is a maximal such independent subset.
- Note that if a set of independent subsets is linearly ordered by inclusion, then their union is also independent.
- The claim then follows from Zorn's Lemma.
- To show that such a maximal independent set must be a basis, we proceed by contradiction:
- If this is not true, there is some vector v that is not in the span.
- Adding this to the set gives us a larger independent set, contradicting maximality.

□

A matroid is a structure that generalizes the notion of linear independence in vector spaces.

We will talk a bit more about matroids next week.

2. Linear Transformations

Definition 3. *Given two vector spaces V and W over a field $\mathbb{F}_b$, recall that a **linear transformation**, or homomorphism, $T : V \to W$ satisfies the following two conditions:*

- *$T(v_1 + v_2) = T(v_1) + t(v_2)$ for all $v_1, v_2 \in V$*
- *$T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{F}$.*

One thing to note is that this immediately extends to finite linear combinations of vectors. Also observe that $T(\mathbf{0}) = \mathbf{0}$.

We denote the set of all transformations from V to W by $\mathrm{Hom}_{\mathbb{F}}(V, W)$. In the case that $V = W$, we denote this by $\mathrm{End}(V)$, and in the case that all maps are injective and surjective, we denote this by $\mathrm{Aut}(V)$. When $V = \mathbb{F}^n$ for some natural number n, we write $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^n) = GL_n(\mathbb{F})$.

Note that the determinant is the unique ring homomorphism between $GL_n(\mathbb{F}) \to \mathbb{F}^{\times}$.

3. Dual Spaces

Let R be a commutative ring. Then, given two R-modules M and N, we may consider the set $\mathrm{Hom}_R(M, N)$ of all R-linear maps from M to N. With the usual notions of addition and scaling, we may endow $\mathrm{Hom}_R(M, N)$ with a ring structure.
Definition: For a vector space V, we call $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ its **dual space**.

Dual spaces are needed to discuss coordinate functions on vector spaces, formalize notions of integration in functional analysis, classify linear transformations via tensor products, and study tangent bundles of smooth manifolds.

Theorem 2. *Given a finite dimensional vector space V, dim$(V) =$ dim(V^*).*

Proof. Choose a basis $\{v_1, \ldots, v_n\}$ of V, and consider the set $\{v^1, \ldots, v^n\}$ of V^*, where $v^i(v_j) := \delta_{ij}$ is 1 if $i = j$ and 0 otherwise. Of course, a linear transformation is determined completely by the values of elements of a basis of its domain. So any linear transformation $f \in V^*$ satisfies

$$f(v) = f\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i f(v_i) \implies f = \sum_{i=1}^{n} a_i f(v_i) v^i,$$

where $v = \sum_{i=1}^{n} a_i v_i$ is any element of V ($a_i \in \mathbb{F}$ for each $i \in \{1, \ldots, n\}$). Thus, $\{v^1, \ldots, v^n\}$ spans V^*. Further, given any nonzero $f = a_1 v^1 + \cdots + a_n v^n$, some a_i is nonzero, so $f(v_i) \neq 0$ implies $f \neq 0$. Therefore, $\{v^1, \ldots, v^n\}$ is linearly independent. The result follows. $\square$

Now, suppose V and W are vector spaces over $\mathbb{F}$, and consider $T \in \mathrm{Hom}_{\mathbb{F}}(V, W)$. Then, we wonder if we can create a natural correspondence somehow between $T : V \to W$ and a linear transformation of V^* and W^*. But given an element $f \in V^*$, $f : V \to \mathbb{F}_b$, if we wish to associate some $g \in W^*$, $g : W \to \mathbb{F}_b$, the only way to make function composition work is to map $f \mapsto f \circ T^{-1}$. However, T^{-1} is not always defined, so it turns out that the map suggested is $T^* : W \to V$ given by $g \mapsto T \circ g$.

4. Free Vector Space

Given a set X and vector space V over field $\mathbb{F}_b$, we may define the **free space** $V\langle X\rangle \subset \mathrm{Hom}_{\mathbb{F}}(X, V)$ such that all $f \in V\langle X\rangle$ satisfy $f(x) = 0$ for all but finitely many $x \in X$. Addition and scalar multiplication are defined as usual.

A crucial example of the free space is a vector field. On a manifold, a vector field is a mapping from the underlying set into the tangent bundle such that the natural projection from the tangent space to the manifold composed with the vector field is the identity.

For a specific example, we consider a set X of 5 points in $\mathbb{R}^2_b$ and the usual vector space $\mathbb{R}^2$. To each point, we assign a "direction".

What does a basis look like? Consider the Dirac delta functions.

Let V be a vector space over $\mathbb{F}$ and X a set. Let $\delta : X \to \mathbb{F}_b\langle X\rangle$ be defined to be $\delta(x) := \delta_x$. This is the Dirac map. It is injective but not linear.

Theorem 3. *Let V be a vector space over a field $\mathbb{F}$ and X a set. If $g : X \to V$ is a function, there exists a unique homomorphism $\widehat{g} : \mathbb{F}_b\langle X\rangle \to V$ such that $g = \widehat{g} \circ \delta$.*

Proof. Note that $\{\delta_x\}_{x\in X}$ is a basis of $\mathbb{F}\langle X\rangle$ and so there exists a unique linear transformation (homomorphism) $\mathbb{F}\langle X\rangle \to V$ mapping $\delta_x \mapsto g(x)$ for each x. $\square$